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# DIMENSIONS OF SLOWLY ESCAPING SETS AND ANNULAR ITINERARIES FOR EXPONENTIAL FUNCTIONS

D. J. SIXSMITH

ABSTRACT. We study the iteration of functions in the exponential family. We construct a number of sets, consisting of points which escape to infinity ‘slowly’, and which have Hausdorff dimension equal to 1. We prove these results by using the idea of an *annular itinerary*. In the case of a general transcendental entire function we show that one of these sets, the *uniformly slowly escaping set*, has strong dynamical properties and we give a necessary and sufficient condition for this set to be non-empty.

## 1. INTRODUCTION

This paper is principally concerned with transcendental entire functions in the *exponential family*, defined by

$$E_\lambda(z) = \lambda e^z, \quad \text{for } \lambda \in \mathbb{C} \setminus \{0\}.$$

For a general transcendental entire function  $f$ , the *Fatou set*  $F(f)$  is defined as the set  $z \in \mathbb{C}$  such that  $\{f^n\}_{n \in \mathbb{N}}$  is a normal family in a neighbourhood of  $z$ . The *Julia set*  $J(f)$  is the complement in  $\mathbb{C}$  of  $F(f)$ . An introduction to the properties of these sets was given in [1]. The *escaping set*, which was first studied for a general transcendental entire function in [10], is defined by

$$I(f) = \{z : f^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty\}.$$

Many authors have studied the Hausdorff dimension of  $I(E_\lambda)$ , or subsets of this set. We refer to [13] for a definition of Hausdorff dimension, which we denote here by  $\dim_H$ . A key result is that of McMullen [17], who showed that  $\dim_H J(E_\lambda) = 2$ . It is well-known that it follows from his construction that  $\dim_H I(E_\lambda) = 2$ .

When  $\lambda \in (0, e^{-1})$  it is known – see [8, 9] – that  $J(E_\lambda)$  consists of an uncountable set of unbounded curves known as a *Cantor bouquet*, and that each curve, except possibly for its finite endpoint, lies in  $I(E_\lambda)$ . In two celebrated papers [14, 15] Karpińska proved the paradoxical fact that the set consisting of these curves excluding their finite endpoints has Hausdorff dimension 1, whereas the set of finite endpoints has Hausdorff dimension 2. A somewhat related result is that of Karpińska and Urbanski [16] who defined subsets of  $I(E_\lambda)$  of Hausdorff dimension  $d$ , for each  $d \in (1, 2)$ .

In fact, all these papers considered a subset  $A(E_\lambda)$  of  $I(E_\lambda)$  known as the *fast escaping set*. The fast escaping set was introduced in [3], and can be defined [22] for a general transcendental entire function  $f$  by

$$(1.1) \quad A(f) = \{z : \text{there exists } \ell \in \mathbb{N} \text{ such that } |f^{n+\ell}(z)| \geq M^n(R, f), \text{ for } n \in \mathbb{N}\}.$$

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Here the *maximum modulus function* is defined by  $M(r, f) = \max_{|z|=r} |f(z)|$ , for  $r \geq 0$ . We write  $M^n(r, f)$  to denote repeated iteration of  $M(r, f)$  with respect to the variable  $r$ . In (1.1),  $R > 0$  is such that  $M^n(R, f) \rightarrow \infty$  as  $n \rightarrow \infty$ .

It is well-known that the sets constructed in [14, 15] and [17] lie in  $A(E_\lambda)$ . We show in Section 7 that this is also the case for the sets defined in [16].

It seems that little is known about the dimension of subsets of  $I(E_\lambda) \setminus A(E_\lambda)$ . Indeed, very little is known about the dimension of  $J(f) \cap (I(f) \setminus A(f))$  for any transcendental entire function  $f$ , with three notable exceptions. Bishop [6] constructed a transcendental entire function  $f_1$  such that  $\dim_H I(f_1) \setminus A(f_1) = 0$ . At the other extreme, Eremenko and Lyubich [11, Example 4] constructed a transcendental entire function  $f_2$  such that  $J(f_2) \cap (I(f_2) \setminus A(f_2))$  has positive area.

The remaining exception concerns the *Eremenko-Lyubich class*,  $\mathcal{B}$ , which is defined as the class of transcendental entire functions for which the set of singular values is bounded. Clearly  $E_\lambda \in \mathcal{B}$ , for  $\lambda \neq 0$ . If  $f \in \mathcal{B}$ , then  $I(f) \subset J(f)$  [12, Theorem 1] and so

$$(1.2) \quad J(f) \cap (I(f) \setminus A(f)) = I(f) \setminus A(f), \quad \text{for } f \in \mathcal{B}.$$

Bergweiler and Peter [5] studied the dimension of subsets of  $I(f)$  consisting of points for which there is a completely general upper bound on the rate of escape. The following is part of [5, Theorem 1].

**Theorem 1.1.** *Suppose that  $f \in \mathcal{B}$  and that  $(p_n)_{n \geq 0}$  is a sequence of real numbers tending to infinity. Define*

$$\text{Esc}(f, (p_n)) = \{z \in I(f) : \text{there exists } N \geq 0 \text{ such that } |f^n(z)| \leq p_n, \text{ for } n \geq N\}.$$

*Then  $\dim_H \text{Esc}(f, (p_n)) \geq 1$ .*

Suppose that  $f \in \mathcal{B}$ . In contrast to Bishop's result, it follows from Theorem 1.1 and (1.2) that  $\dim_H J(f) \cap (I(f) \setminus A(f)) \geq 1$ . Rempe and Stallard [19] showed that there exists a transcendental entire function  $f_3 \in \mathcal{B}$  such that  $\dim_H I(f_3) = 1$ . It follows from Theorem 1.1 and (1.2) that  $\dim_H J(f_3) \cap (I(f_3) \setminus A(f_3)) = 1$ .

In this paper we show that various subsets of  $J(E_\lambda) \cap (I(E_\lambda) \setminus A(E_\lambda))$  have Hausdorff dimension exactly equal to 1. We do not use Theorem 1.1, since all our results give an exact value for the Hausdorff dimension of sets defined by a two-sided inequality on the rate of escape. However, it seems plausible that there is some relationship between these results.

For a general transcendental entire function  $f$  we define the *uniformly slowly escaping set* by

$$(1.3) \quad L_U(f) = \{z : \exists N \in \mathbb{N}, R > 1, 0 < C_1 < C_2 \text{ s.t. } C_1 R^n \leq |f^n(z)| \leq C_2 R^n, \text{ for } n \geq N\}.$$

Roughly speaking, this set consists of those points for which the rate of escape is eventually uniformly slow. Our first result concerns the Hausdorff dimension of  $L_U(E_\lambda)$ .

**Theorem 1.2.** *Suppose that  $\lambda \neq 0$ . Then  $\dim_H L_U(E_\lambda) = 1$ .*

In Section 6 we give, for a general transcendental entire function, a necessary and sufficient condition for the uniformly slowly escaping set to be non-empty. We also prove that when the uniformly slowly escaping set is not empty, it has a number

of familiar properties which show that, in general, this is a dynamically interesting set.

For a general transcendental entire function  $f$ ,  $L_U(f)$  is a subset of the *slow escaping set*, introduced by Rippon and Stallard [21], and defined by

$$(1.4) \quad L(f) = \{z \in I(f) : \text{there exists } R > 1 \text{ s.t. } |f^n(z)| \leq R^n, \text{ for } n \in \mathbb{N}\}.$$

It was shown in [21] that  $L(f) \neq \emptyset$ , that  $J(f)$  is dense in  $L(f)$  and also that  $J(f) = \partial L(f)$ .

It follows from Theorem 1.1 that  $\dim_H L(E_\lambda) \geq 1$ . Nothing more seems to be known about the actual dimension of  $L(E_\lambda)$ . As a step in that direction, we consider, for a general transcendental entire function  $f$ , a set which is a relatively large subset of  $L(f)$  and which contains  $L_U(f)$ . First, for  $p \in \mathbb{N}$ , let  $\log^{+p}$  denote  $p$  iterations of the  $\log^+$  function, which is defined by

$$\log^+(x) = \begin{cases} \log x, & \text{if } x \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

For a general transcendental entire function,  $f$ , we define

$$(1.5) \quad L_A(f) = \{z : \exists R > 1, N, p \in \mathbb{N} \text{ s.t. } n^{\log^{+p}(n)} \leq |f^n(z)| \leq R^n, \text{ for } n \geq N\}.$$

Note that the orbits of points in  $L_A(f)$  are constrained to lie within certain annuli. The fact that

$$L_U(f) \subset L_A(f) \subset L(f) \subset I(f) \setminus A(f)$$

follows from (1.3), (1.4), (1.5) and well-known properties of the maximum modulus function. Our result concerning the dimension of  $L_A(E_\lambda)$  is as follows.

**Theorem 1.3.** *Suppose that  $\lambda \neq 0$ . Then  $\dim_H L_A(E_\lambda) = 1$ .*

Theorem 1.3 is a consequence of the size of the annuli in the definition of  $L_A(E_\lambda)$ . In particular, Theorem 1.3 shows that if  $\dim_H L(E_\lambda) > 1$ , then the vast majority of points in  $L(E_\lambda)$  must have an extremely slowly escaping subsequence.

The techniques that we use to prove Theorem 1.3 also allow us to construct subsets of  $I(E_\lambda) \setminus (L(E_\lambda) \cup A(E_\lambda))$  which have Hausdorff dimension equal to 1. For example, for a transcendental entire function  $f$ , Rippon and Stallard [21] defined the *moderately slow escaping set* by

$$M(f) = \{z \in I(f) : \text{there exists } C > 0 \text{ such that } |f^n(z)| \leq \exp(e^{Cn}), \text{ for } n \in \mathbb{N}\}.$$

In a similar way to (1.5), we define a subset of  $M(f) \setminus L(f)$  by

$$(1.6) \quad M_A(f) = \{z : \exists N, p \in \mathbb{N} \text{ s.t. } e^{n \log^{+p}(n)} \leq |f^n(z)| \leq \exp(e^{pn}), \text{ for } n \geq N\}.$$

Our result concerning the dimension of  $M_A(E_\lambda)$  is as follows.

**Theorem 1.4.** *Suppose that  $\lambda \neq 0$ . Then  $\dim_H M_A(E_\lambda) = 1$ .*

We prove our results using the idea of an annular itinerary. Before defining this concept, we briefly discuss a different type of itinerary which has frequently been used to study the dynamics of functions in the exponential family.

Since  $|E_\lambda(z)| = |\lambda|e^{\operatorname{Re}(z)}$ , it follows that the orbit of a point in  $I(E_\lambda)$  must eventually remain in the right half-plane  $\mathbb{H} = \{z : \operatorname{Re}(z) > 0\}$ . Many authors – see, for example, [7, 8] and [16] – have considered itineraries of points in  $I(E_\lambda)$  defined in the following way. First we partition  $\mathbb{H}$  into half-open strips

$$(1.7) \quad V_n = \{z \in \mathbb{H} : (2n-1)\pi \leq \operatorname{Im}(z) < (2n+1)\pi, \text{ for } n \in \mathbb{Z}\}.$$

Suppose that  $\underline{s} = s_0 s_1 s_2 \dots$  is a sequence of integers. We say that a point  $z$  has *itinerary*  $\underline{s} = \underline{s}(z) = s_0 s_1 s_2 \dots$  if  $E_\lambda^n(z) \in V_{s_n}$ , for  $n \geq 0$ .

For some types of itinerary it can be shown that the set of points with such an itinerary is – in some sense – large. For example, it follows from McMullen’s proof [17] (and see also [14]) that the set

$$\{z \in I(E_\lambda) : 2\pi|s_n(z)| \geq |E_\lambda^n(z)|/2, \text{ for } n \geq 0\}$$

has Hausdorff dimension 2.

The concept of an *annular itinerary* was introduced by Rippon and Stallard [20]. Suppose that  $f$  is a general transcendental entire function, and let  $(R_n)_{n \geq 0}$  be a strictly increasing sequence of positive real numbers such that  $R_n \rightarrow \infty$  as  $n \rightarrow \infty$ . The strips  $V_n$  in (1.7) are replaced by half-open annuli

$$A_n = \{z : R_{n-1} \leq |z| < R_n\}, \quad \text{for } n \in \mathbb{N},$$

and  $A_0$  is defined as  $\{z : |z| < R_0\}$ . Suppose that  $\underline{t} = t_0 t_1 t_2 \dots$  is a sequence of non-negative integers. If  $f^n(z) \in A_{t_n}$ , for  $n \geq 0$ , then we say that the point  $z$  has *annular itinerary*  $\underline{t} = \underline{t}(z) = t_0 t_1 t_2 \dots$  for the function  $f$  with respect to the partition  $(A_n)_{n \geq 0}$ .

Rippon and Stallard [20] let  $R_0 > 0$  be sufficiently large that  $M^n(R_0, f) \rightarrow \infty$  as  $n \rightarrow \infty$ , and then set  $R_n = M^n(R_0, f)$ , for  $n \in \mathbb{N}$ . They showed that, with this choice of partition  $(A_n)_{n \geq 0}$ , there is a very broad class of annular itineraries such that the set of points with such an itinerary contains a point in  $J(f)$ . For more information regarding the properties of these annular itineraries, we refer to [20].

Annular itineraries are a natural choice when studying points which escape to infinity with different rates. The annular itineraries used in our paper are defined using annuli of constant modulus, which seems a natural choice when considering points in the slow escaping set. First we choose a value of  $R > 1$ , and then set  $R_n = R^{n+1}$ , for  $n \geq 0$ . This construction of the partition  $(A_n)_{n \geq 0}$  should be considered to be in place throughout the remainder of this paper. Note that this construction depends on  $R$ . Here, and elsewhere, we suppress some dependencies for simplicity of notation, and retain only dependencies which need to remain explicit.

We use the following notation

$$I_R(\underline{t}) = \{z : z \text{ has annular itinerary } \underline{t} \text{ with respect to the partition } (A_n)_{n \geq 0}\}.$$

Note that though this definition concerns an arbitrary map  $f$ , in this paper we are only concerned with the exponential family.

We are interested in a particular type of annular itinerary. We say that an annular itinerary  $\underline{t} = t_0 t_1 t_2 \dots$  is *non-zero* if  $t_n \neq 0$ , for  $n \geq 0$ , *escaping* if  $t_n \rightarrow \infty$  as  $n \rightarrow \infty$ , *admissible* if  $e^{t_n} > t_{n+1}$ , for  $n \geq 0$ , and *slowly-growing* if

$$(1.8) \quad \lim_{n \rightarrow \infty} \frac{t_n}{\sum_{k=1}^{n-1} t_k} = 0.$$

Our main result regarding annular itineraries is as follows.

**Theorem 1.5.** *Suppose that  $\lambda \neq 0$ ,  $R > 1$  and  $\underline{t}$  is an escaping annular itinerary for the function  $E_\lambda$ . Then  $\dim_H I_R(\underline{t}) \leq 1$ . Moreover, there exists  $R_0 = R_0(\lambda) > 1$  such that if, in addition,  $R \geq R_0$  and  $\underline{t}$  is non-zero, admissible and slowly-growing, then  $\dim_H I_R(\underline{t}) = 1$ .*

**Remark 1.** It seems surprising that, for a large class of annular itineraries, the sets of points with the same annular itinerary all have the same Hausdorff dimension. We note that there are annular itineraries of arbitrarily slow growth which satisfy the conditions of Theorem 1.5. In other words, if  $(p_n)_{n \geq 0}$  is a sequence of positive integers such that  $p_n \rightarrow \infty$  as  $n \rightarrow \infty$ , then there exists an annular itinerary  $\underline{t} = t_0 t_1 t_2 \dots$  and  $R > 1$  such that  $\dim_H I_R(\underline{t}) = 1$  and  $t_n \leq p_n$ , for  $n \geq 0$ .

**Remark 2.** We comment briefly on the final two conditions in the second part of Theorem 1.5. The condition that the annular itinerary be admissible is sufficient to ensure that  $R$  may be chosen large enough (depending on  $\lambda$ ) such that  $I_R(\underline{t})$  is not empty. It is unclear if the condition (1.8) is essential, though it is required for our method of proof. It is a straightforward calculation to show that a sequence  $(t_n)_{n \in \mathbb{N}}$  which satisfies this condition also satisfies

$$(1.9) \quad \log t_n = o(n) \text{ as } n \rightarrow \infty.$$

However, the condition (1.9) is weaker than the condition (1.8). For example, consider the sequence defined by

$$t_n = 2^{m^2}, \quad \text{for } (m-1)^3 \leq n < m^3, \quad m \in \mathbb{N}.$$

It can be shown that this sequence satisfies (1.9) but not (1.8). The techniques of this paper do not allow us to replace (1.8) with the apparently simpler condition (1.9).

Finally, we note that the dimension of subsets of  $J(E_\lambda)$  which lie outside of  $I(E_\lambda)$  was studied in [14, Theorem 2] and [25]. In addition, Pawelec and Zdunik [18] recently showed that, for certain values of  $\lambda$ , there exist indecomposable continua in  $J(E_\lambda)$  which are of Hausdorff dimension 1. These continua intersect with the fast escaping set. We refer to [18] for further details.

The structure of this paper is as follows. First, in Section 2, we give some preliminary lemmas. In Section 3 we prove a theorem which gives a lower bound on the Hausdorff dimension of  $I_R(\underline{t})$  for a certain type of annular itinerary. In Section 4 we prove a theorem which gives an upper bound on the Hausdorff dimension of certain sets. All our dimension results are consequences of these two theorems. In Section 5 we prove Theorem 1.2, Theorem 1.3, Theorem 1.4 and Theorem 1.5. In Section 6, we state and prove two results about the uniformly slowly escaping set. Finally, in Section 7, we discuss, briefly, the result of Karpińska and Urbański mentioned earlier.

## 2. PRELIMINARY LEMMAS

We start this section with two lemmas concerning functions in the exponential family. We define closed annuli and half-annuli, for  $0 < r_1 < r_2$ , by

$$(2.1) \quad A(r_1, r_2) = \{z : r_1 \leq |z| \leq r_2\} \text{ and } H(r_1, r_2) = \{z \in A(r_1, r_2) : \operatorname{Re}(z) \geq 0\}.$$

For  $r > 0$  and  $a \in \mathbb{C}$ , we write  $B(a, r)$  for the open disc  $\{z : |z - a| < r\}$ .

The first lemma provides an estimate on the density of preimages of one half-annulus in another; see Figure 1. Here, for measurable sets  $U$  and  $V$ , we define

$$\operatorname{dens}(U, V) = \frac{\operatorname{area}(U \cap V)}{\operatorname{area}(V)},$$

where  $\operatorname{area}(U)$  denotes the Lebesgue measure of  $U$ .

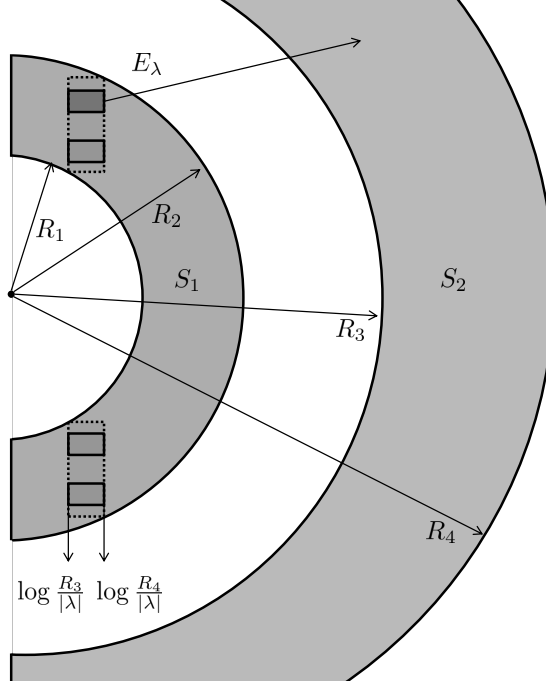


FIGURE 1. The set  $E_\lambda^{-1}(S_2) \cap S_1$ . One preimage component of  $S_2$  is shown with a slightly darker background. The two rectangles constructed in the proof of Lemma 2.1 are shown with a dashed boundary. Note that  $R_3$  is not necessarily larger than  $R_2$ .

**Lemma 2.1.** *Suppose that  $0 < R_1 < R_2$  and  $0 < R_3 < R_4$  are such that*

$$(2.2) \quad R_2 > \max \left\{ 2R_1, R_1 + 16\pi, 3 \log \frac{R_4}{|\lambda|} \right\},$$

*and*

$$(2.3) \quad R_3 > |\lambda|.$$

*Let  $S_1 = H(R_1, R_2)$ ,  $S_2 = H(R_3, R_4)$ , and let  $D$  be the union of all the components of  $E_\lambda^{-1}(S_2)$  which are contained in  $S_1$ . Then*

$$(2.4) \quad \text{dens}(D, S_1) \geq \frac{1}{2\pi R_2} \log \frac{R_4}{R_3}.$$

*Proof.* Each component of  $E_\lambda^{-1}(S_2)$  is a rectangle of the form, for  $n \in \mathbb{Z}$ ,

$$(2.5) \quad \left\{ z : \log \frac{R_3}{|\lambda|} \leq \text{Re}(z) \leq \log \frac{R_4}{|\lambda|}, \left( 2n - \frac{1}{2} \right) \pi \leq \text{Im}(z) + \arg(\lambda) \leq \left( 2n + \frac{1}{2} \right) \pi \right\}.$$

Suppose that the inequalities (2.2) and (2.3) both hold. Consider two large rectangles, each with sides parallel to the coordinate axes. One rectangle has a vertex at the point in the upper half-plane where the vertical line  $\{z : \text{Re}(z) = \log \frac{R_3}{|\lambda|}\}$  meets the circle  $B(0, R_1)$ ; note that if  $R_1 \leq \log \frac{R_3}{|\lambda|}$  we put this vertex at  $R_1$ . The diagonally opposite vertex of this rectangle is at the point in the upper half-plane

where the vertical line  $\{z : \operatorname{Re}(z) = \log \frac{R_4}{|\lambda|}\}$  meets the circle  $B(0, R_2)$ . The second rectangle is the complex conjugate of the first one.

Let  $h$  be the height of each rectangle. It follows by an application of Pythagoras's theorem to this rectangle, and by (2.2), that

$$\begin{aligned} h &= \left( R_2^2 - \left( \log \frac{R_4}{|\lambda|} \right)^2 \right)^{\frac{1}{2}} - \left( \max \left\{ 0, R_1^2 - \left( \log \frac{R_3}{|\lambda|} \right)^2 \right\} \right)^{\frac{1}{2}} \\ &\geq \frac{7}{8} R_2 - R_1 \geq \frac{3}{4} (R_2 - R_1). \end{aligned}$$

It follows by (2.2) and (2.5) that each rectangle contains at least  $\frac{1}{4\pi}(R_2 - R_1)$  components of  $E_\lambda^{-1}(S_2)$ . Hence  $S_1$  contains at least  $\frac{1}{2\pi}(R_2 - R_1)$  components of  $E_\lambda^{-1}(S_2)$ , each of which is a closed rectangle of height  $\pi$  and width at least  $\log \frac{R_4}{R_3}$ . Equation (2.4) follows from this, and the fact that  $\operatorname{area}(S_1) = \frac{1}{2}\pi(R_2^2 - R_1^2)$ .  $\square$

For a domain  $V$  and a transcendental entire function  $f$ , univalent in  $V$ , we define the *distortion* of  $f$  in  $V$  by

$$(2.6) \quad D_V(f) = \frac{\sup_{z \in V} |f'(z)|}{\inf_{z \in V} |f'(z)|}.$$

For functions in the exponential family, the following facts are immediate.

**Lemma 2.2.** *Suppose that  $F$  is a set such that  $\inf\{|z| : z \in F\} = r_1 > 0$  and  $\sup\{|z| : z \in F\} = r_2$ , and that  $V$  is a component of  $E_\lambda^{-1}(F)$  such that  $E_\lambda$  is univalent in  $V$ . Then*

$$(2.7) \quad |E'_\lambda(z)| \geq r_1, \quad \text{for } z \in V,$$

and

$$(2.8) \quad D_V(E_\lambda) = \frac{r_2}{r_1}.$$

We also use two well-known properties of Hausdorff dimension. For the first see, for example, [13].

**Lemma 2.3.** *Suppose that  $(F)_{i \in I}$  is a collection of subsets of  $\mathbb{C}$ , and that  $I$  is a finite or countable set. Then*

$$\dim_H \bigcup_{i \in I} F_i = \sup_{i \in I} \{\dim_H F_i\}.$$

The second property is used frequently but we are not aware of a reference.

**Lemma 2.4.** *Suppose that  $f$  is a non-constant transcendental entire function and that  $U \subset \mathbb{C}$ . Then*

$$\dim_H f(U) = \dim_H f^{-1}(U) = \dim_H U.$$

### 3. A LOWER BOUND ON THE HAUSDORFF DIMENSION

In this section we prove the following theorem which gives a lower bound on the Hausdorff dimension of  $I_R(\underline{t})$  for a certain type of annular itinerary.

**Theorem 3.1.** *Suppose that  $\lambda \neq 0$ . Then there exists  $R_0 = R_0(\lambda) > 1$  such that, if  $R \geq R_0$  and  $\underline{t}$  is an escaping, non-zero, admissible and slowly-growing annular itinerary for the function  $E_\lambda$ , then  $\dim_H I_R(\underline{t}) \geq 1$ .*



To prove Theorem 3.1, we use a well-known construction and result of McMullen. Let  $(\mathcal{E}_n)_{n \geq 0}$  be a sequence of finite collections of pairwise disjoint compact subsets of  $\mathbb{C}$  such that the following both hold:

- (i) If  $F \in \mathcal{E}_{n+1}$ , then there exists a unique  $G \in \mathcal{E}_n$  such that  $F \subset G$ ;
- (ii) If  $G \in \mathcal{E}_n$ , then there exists at least one  $F \in \mathcal{E}_{n+1}$  such that  $G \supset F$ .

We write

$$(3.1) \quad D_n = \bigcup_{F \in \mathcal{E}_n} F, \text{ for } n \geq 0, \quad \text{and} \quad D = \bigcap_{n \geq 0} D_n.$$

McMullen's result is the following [17, Proposition 2.2]. Here, for a set  $U$ ,  $\text{diam } U$  denotes the Euclidean diameter of  $U$ .

**Lemma 3.1.** *Suppose that there exists a sequence of finite collections of pairwise disjoint compact sets,  $(\mathcal{E}_n)_{n \geq 0}$ , which satisfies conditions (i) and (ii) above, and let  $D$  and  $(D_n)_{n \geq 0}$  be as defined in (3.1). Suppose also that  $(\Delta_n)_{n \geq 0}$  and  $(d_n)_{n \geq 0}$  are sequences of positive real numbers, with  $d_n \rightarrow 0$  as  $n \rightarrow \infty$ , such that, for each  $n \geq 0$  and for each  $F \in \mathcal{E}_n$ , we have*

$$\text{dens}(D_{n+1}, F) \geq \Delta_n \quad \text{and} \quad \text{diam } F \leq d_n.$$

Then

$$(3.2) \quad \dim_H D \geq 2 - \limsup_{n \rightarrow \infty} \frac{\sum_{m=0}^n |\log \Delta_m|}{|\log d_n|}.$$

**Remark 3.** In [17] the upper bound of summation in (3.2) was given as  $n + 1$ . However, the stronger result (3.2) – which is required in the proof of Theorem 3.1 – follows from McMullen's proof, and has been given in, for example, [4, Lemma 4.4] and [24, Lemma 4.3].

We also use the following. This is a version of [23, Lemma 5.2], which itself is a detailed version of [17, Proposition 3.1].

**Lemma 3.2.** *Suppose that  $f$  is a transcendental entire function, and there exists a set  $U \subset \mathbb{C}$  and constants  $\alpha > 1$  and  $M > 0$  such that*

$$(3.3) \quad |f'(z)| > \alpha \quad \text{and} \quad \left| \frac{f''(z)}{f'(z)} \right| < M, \quad \text{for } z \in U.$$

*Suppose also that there exists  $s \in (0, (4M)^{-1})$  such that if  $B \subset U$  is a disc of diameter  $s$ , then  $f$  is conformal in a neighbourhood of  $B$ . Suppose finally that  $(B_m)_{m \in \{1, 2, \dots, n\}}$  is a sequence of sets contained in  $U$ , each of diameter less than  $s$ , and such that*

$$B_{m+1} \subset f(B_m), \quad \text{for } m \in \{1, 2, \dots, n-1\}.$$

*For  $m \in \{1, 2, \dots, n\}$ , let  $\phi_m$  be the inverse branch of  $f$  which maps  $f(B_m)$  to  $B_m$ , and set  $V = \phi_1 \circ \phi_2 \circ \dots \circ \phi_n(f(B_n))$ . Then there exists  $\tau = \tau(M, s, \alpha) > 1$  such that*

$$D_V(f^n) \leq \tau.$$

Note that in [23, Lemma 5.2] the sets  $B_n$  are squares of side  $s$ . The proof of the above result follows in exactly the same way, and is omitted.

We deduce the following.

**Corollary 3.1.** *There exist absolute constants  $s_0 > 0$  and  $\tau_0 > 1$  such that the following holds. Suppose that  $\lambda \neq 0$ ,  $n \in \mathbb{N}$  and  $V$  is a set such that*

$$E_\lambda^m(V) \subset \left\{ z : \operatorname{Re}(z) > \log \frac{2}{|\lambda|} \right\} \text{ and } \operatorname{diam} E_\lambda^m(V) < s_0, \quad \text{for } 0 \leq m < n.$$

*Then  $D_V(E_\lambda^n) \leq \tau_0$ .*

*Proof.* This result follows from Lemma 3.2 with  $f = E_\lambda$ ,  $U = \left\{ z : \operatorname{Re}(z) > \log \frac{2}{|\lambda|} \right\}$ ,  $\alpha = M = 2$ , and  $B_m = E_\lambda^{m-1}(V)$ , for  $1 \leq m \leq n$ .  $\square$

We now give the proof of Theorem 3.1. Roughly speaking, our method of proof is as follows. First we set a value of  $R_0$  sufficiently large to enable us to use Lemma 2.1. We then define a set which is contained in  $I_R(\underline{t})$  and apply McMullen's result to obtain a lower bound on the Hausdorff dimension of this set.

*Proof of Theorem 3.1.* Let  $s_0$  be the constant in Corollary 3.1. We choose

$$(3.4) \quad R_0 > \max \left\{ e, |\lambda|, \frac{2}{s_0} \right\},$$

sufficiently large that

$$(3.5) \quad R(R-1) > 16\pi + 2 \quad \text{and} \quad R > 3 \log \frac{R^2}{|\lambda|} + 1, \quad \text{for } R \geq R_0.$$

Suppose that  $R \geq R_0$ , and that  $\underline{t}$  is an escaping, non-zero, admissible and slowly-growing annular itinerary. To use Lemma 3.1 we need to work with compact and disjoint sets. In order to do this, and recalling the definition (2.1), we define disjoint closed half-annuli

$$H_n = H(R^n + 1, R^{n+1} - 1), \quad \text{for } n \in \mathbb{N}.$$

Since  $\underline{t}$  is admissible and non-zero, we deduce by (3.4) and (3.5) that, for  $n \geq 0$ , we have

$$(3.6) \quad R^{t_{n+1}} > R e^{t_n} > R t_{n+1} > 3 t_{n+1} \log R + 3 \log \frac{R}{|\lambda|} + 1 > 3 \log \left( \frac{R^{t_{n+1}+1} - 1}{|\lambda|} \right) + 1.$$

Since  $\underline{t}$  is non-zero, we deduce from (3.4), (3.5) and (3.6) that the hypotheses of Lemma 2.1 are satisfied with  $S_1 = H_{t_n}$  and  $S_2 = H_{t_{n+1}}$ , for  $n \geq 0$ .

In order to use Lemma 3.1, we define a sequence of finite collections of pairwise disjoint compact sets as follows. First set

$$\mathcal{E}_0 = \{H_{t_0}\},$$

and, for  $n \geq 0$ ,

$$\mathcal{E}_{n+1} = \{F : F \subset G, \text{ for some } G \in \mathcal{E}_n, \text{ and } E_\lambda^{n+1}(F) = H_{t_{n+1}}\}.$$

Let  $D_n$ , for  $n \geq 0$ , and  $D$  be the sets defined in (3.1). It follows from (3.1) that  $D \subset I_R(\underline{t})$ . It is sufficient, therefore, to show that  $\dim_H D \geq 1$ .

It follows from Lemma 2.1 that the conditions (i) and (ii) stated prior to Lemma 3.1 are both satisfied. It remains to estimate the diameters and the densities stated in Lemma 3.1.

Since  $\underline{t}$  is escaping, we can let  $N_0 \geq 2$  be sufficiently large that  $t_{n-1} \geq 2$ , for  $n \geq N_0$ . Suppose that  $n \geq N_0$  and that  $F \in \mathcal{E}_n$ . Note that  $E_\lambda^n(F) = H_{t_n}$ . We first find an upper bound on the diameter of  $F$ . Since

$$\text{diam } E_\lambda^n(F) = \text{diam } H_{t_n} < 2R^{t_n+1},$$

we have, by (2.7), that

$$(3.7) \quad \text{diam } F \leq 2R^{t_n+1} \frac{1}{R^{t_1}} \frac{1}{R^{t_2}} \cdots \frac{1}{R^{t_n}} = 2R^{1-\sum_{m=1}^{n-1} t_m}.$$

We set  $d_n = 2R^{1-\sum_{m=1}^{n-1} t_m}$ , for  $n \geq 0$ . Since  $\underline{t}$  is escaping, we deduce that

$$(3.8) \quad d_n \rightarrow 0 \quad \text{and} \quad |\log d_n| = \log R \sum_{m=1}^{n-1} t_m (1 + o(1)) \quad \text{as } n \rightarrow \infty.$$

We next show that the distortion of  $E_\lambda^n$  on  $F$  is bounded independently of  $n$  and  $F$ . Once again by (2.7), and by (3.4), we have

$$\text{diam } E_\lambda^m(F) \leq 2R^{t_n+1} \frac{1}{R^{t_{n-1}}} \frac{1}{R^{t_n}} \leq \frac{2}{R} < s_0, \quad \text{for } 0 \leq m < n-1.$$

Suppose that  $0 \leq m < n-1$  and that  $z \in E_\lambda^m(F)$ . Since  $\underline{t}$  is non-zero, we have  $|E_\lambda(z)| \geq R$ . We deduce by (3.4) that

$$\text{Re}(z) \geq \log \frac{R}{|\lambda|} > \log \frac{2}{|\lambda|}.$$

Hence

$$E_\lambda^m(F) \subset \left\{ z : \text{Re}(z) > \log \frac{2}{|\lambda|} \right\}, \quad \text{for } 0 \leq m < n-1.$$

We deduce by Corollary 3.1 that  $D_F(E_\lambda^{n-1}) \leq \tau_0$ . Moreover, it follows from (2.8) that  $D_{E_\lambda^{n-1}(F)}(E_\lambda) \leq R$ . Thus

$$(3.9) \quad D_F(E_\lambda^n) \leq D_F(E_\lambda^{n-1}) D_{E_\lambda^{n-1}(F)}(E_\lambda) \leq \tau_0 R.$$

We use (3.9) to find a lower bound on  $\text{dens}(D_{n+1}, F)$ . Note that  $E_\lambda^n(D_{n+1})$  consists of those components of  $E_\lambda^{-1}(H_{t_{n+1}})$  which are contained in  $H_{t_n}$ . Hence, by (2.4), (3.4) and (3.9), we have

$$\begin{aligned} \text{dens}(D_{n+1}, F) &\geq \frac{1}{(\tau_0 R)^2} \text{dens}(E_\lambda^n(D_{n+1}), E_\lambda^n(F)) \\ &\geq \frac{1}{(\tau_0 R)^2} \frac{1}{2\pi(R^{t_n+1}-1)} \log \frac{R^{t_{n+1}+1}-1}{R^{t_n+1}+1} \\ &\geq \frac{\log R}{4\tau_0^2 \pi R^{t_n+3}}. \end{aligned}$$

We set  $\Delta_n = \frac{\log R}{4\tau_0^2 \pi R^{t_n+3}}$ , for  $n \geq 0$ . Note that

$$(3.10) \quad |\log \Delta_n| = t_n \log R (1 + o(1)) \quad \text{as } n \rightarrow \infty.$$

Since  $\underline{t}$  is escaping and slowly-growing, it follows, by (1.8), (3.8) and (3.10), that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\sum_{m=0}^n |\log \Delta_m|}{|\log d_n|} &= \limsup_{n \rightarrow \infty} \frac{\log R \sum_{m=1}^n t_m(1+o(1))}{\log R \sum_{m=1}^{n-1} t_m(1+o(1))} \\ &= \limsup_{n \rightarrow \infty} \left( \frac{t_n(1+o(1))}{\sum_{m=1}^{n-1} t_m(1+o(1))} + \frac{\sum_{m=1}^{n-1} t_m(1+o(1))}{\sum_{m=1}^{n-1} t_m(1+o(1))} \right) \\ &= 1. \end{aligned}$$

We deduce by Lemma 3.1 that  $\dim_H D \geq 1$ , as required.  $\square$

#### 4. AN UPPER BOUND ON HAUSDORFF DIMENSION

In this section we prove a theorem which gives an upper bound on the Hausdorff dimension of certain sets, and so is, in a sense, complementary to Theorem 3.1. First we define the sets. Suppose that, for each  $p \in \mathbb{N}$ ,  $(g_{p,n})_{n \in \mathbb{N}}$  and  $(h_{p,n})_{n \in \mathbb{N}}$  are sequences of positive real numbers such that

$$(4.1) \quad h_{p,n} \geq g_{p,n}, \quad \text{for } n \in \mathbb{N}.$$

For  $\lambda \neq 0$  define the set  $T_{g,h}$  by

$$(4.2) \quad T_{g,h} = \{z : \text{there exist } N, p \in \mathbb{N} \text{ s.t. } E_\lambda^n(z) \in A(g_{p,n}, h_{p,n}), \text{ for } n \geq N\}.$$

Our theorem is as follows.

**Theorem 4.1.** *Suppose that  $\lambda \neq 0$ , and that for each  $p \in \mathbb{N}$ ,  $(g_{p,n})_{n \in \mathbb{N}}$  and  $(h_{p,n})_{n \in \mathbb{N}}$  are sequences of positive real numbers such that (4.1) is satisfied,*

$$(4.3) \quad g_{p,n} \rightarrow \infty \text{ as } n \rightarrow \infty,$$

and

$$(4.4) \quad \frac{\log g_{p,n}}{\log^+ \log h_{p,n+1}} \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Then  $\dim_H T_{g,h} \leq 1$ .

Note that the condition (4.4) ensures that successive iterates lie in annuli which are, in a sense, not ‘too thick’. It is not related to the condition of admissibility, which is used to obtain a lower bound on Hausdorff dimension.

*Proof of Theorem 4.1.* Choose

$$(4.5) \quad \beta > \max \left\{ \sqrt{1 + \pi^2}, \frac{1}{s_0} \right\}$$

and set

$$c_0 = |\lambda| e^{1+\beta} \max \left\{ \frac{2}{|\lambda|}, \exp(\beta^2) \right\}$$

and

$$(4.6) \quad c_1 = \log \frac{c_0}{|\lambda|} = 1 + \beta + \max \left\{ \log \frac{2}{|\lambda|}, \beta^2 \right\},$$

where  $s_0$  is the constant in Corollary 3.1. By (4.3), for each  $p \in \mathbb{N}$  we can choose  $\ell_p \in \mathbb{N}$  such that

$$(4.7) \quad g_{p,n} > c_0, \quad \text{for } n \geq \ell_p.$$

Define sets

$$(4.8) \quad S_{\nu,p} = \{z : E_\lambda^n(z) \in A(g_{p,\nu+n}, h_{p,\nu+n}), \text{ for } n \geq 0\}, \quad \text{for } \nu, p \in \mathbb{N}.$$

Fix a value of  $p \in \mathbb{N}$ , and suppose that  $\nu \geq \ell_p$ . We claim that

$$(4.9) \quad \dim_H S_{\nu,p} \leq 1.$$

Before proving (4.9) we show that the result of the lemma can be deduced from this equation. First we claim that

$$(4.10) \quad T_{g,h} \subset \bigcup_{p \in \mathbb{N}} \bigcup_{\nu \geq \ell_p} E_\lambda^{-\nu}(S_{\nu,p}).$$

For, suppose that  $z \in T_{g,h}$ , in which case there exist  $N, p \in \mathbb{N}$  such that

$$E_\lambda^n(z) \in A(g_{p,n}, h_{p,n}), \quad \text{for } n \geq N.$$

It follows that there exists  $\nu \geq \max\{N, \ell_p\}$  such that  $E_\lambda^n(E_\lambda^\nu(z)) \in A(g_{p,n+\nu}, h_{p,n+\nu})$ , for  $n \geq 0$ , and so  $E_\lambda^\nu(z) \in S_{\nu,p}$ . This establishes equation (4.10). Theorem 4.1 follows from (4.9) and (4.10), by Lemma 2.3 and Lemma 2.4.

If remains to show that (4.9) holds for  $\nu \geq \ell_p$ . We establish this result using a sequence of covers of  $S_{\nu,p}$  and basic properties of Hausdorff dimension. We suppress the variable  $p$  for simplicity.

First we note some properties of  $S_\nu$  and then use these properties to define a sequence of covers of this set. It follows from (4.7), and the fact that  $\nu \geq \ell$ , that

$$|E_\lambda(E_\lambda^n(z))| = |\lambda| e^{\operatorname{Re}(E_\lambda^n(z))} > c_0, \quad \text{for } z \in S_\nu, \quad n \geq 0.$$

We deduce by (4.6) that

$$(4.11) \quad \operatorname{Re}(E_\lambda^n(z)) > \log \frac{c_0}{|\lambda|} = c_1, \quad \text{for } z \in S_\nu, \quad n \geq 0.$$

Define compact sets

$$(4.12) \quad W_{n,m} = A(e^{m-1}g_n, e^m g_n) \cap \{z : \operatorname{Re}(z) \geq c_1\}, \quad \text{for } n, m \in \mathbb{N}.$$

We observe that each component of  $E_\lambda^{-1}(W_{n,m})$ , for  $m, n \in \mathbb{N}$ , is contained in a distinct rectangle of the form,

$$(4.13) \quad \{z : -1 \leq \operatorname{Re}(z) - m - \log g_n + \log |\lambda| \leq 0, \quad -\pi/2 \leq \operatorname{Im}(z) + \arg(\lambda) + 2k\pi \leq \pi/2\},$$

for some  $k \in \mathbb{Z}$ . Hence, by (4.5), if  $F$  is a component of  $E_\lambda^{-1}(W_{n,m})$ , then

$$(4.14) \quad \operatorname{diam} F \leq \beta, \quad \text{for } m, n \in \mathbb{N}.$$

For simplicity of notation, define sets of integers

$$(4.15) \quad \alpha_n = \left\{1, 2, \dots, \left\lceil \log \frac{h_n}{g_n} \right\rceil + 1\right\}, \quad \text{for } n \in \mathbb{N},$$

where  $[x]$  denotes the integer part of  $x$ . Note that

$$(4.16) \quad A(g_n, h_n) \cap \{z : \operatorname{Re}(z) \geq c_1\} \subset \bigcup_{m \in \alpha_n} W_{n,m}, \quad \text{for } n \in \mathbb{N}.$$

Now set

$$\mathcal{E}_0 = \{W_{\nu,m} : m \in \alpha_\nu\},$$

and, for  $n \geq 0$ ,

$$\mathcal{E}_{n+1} = \{F : E_\lambda^{n+1}(F) = W_{\nu+n+1,m} \text{ for some } m \in \alpha_{\nu+n+1} \text{ and } F \cap G \neq \emptyset \text{ for some } G \in \mathcal{E}_n\}.$$

It follows by (4.11) and (4.16) that

$$S_\nu \subset \bigcup_{F \in \mathcal{E}_n} F, \quad \text{for each } n \geq 0.$$

This completes the definition of the sequence of covers of  $S_\nu$ .

Next we study the properties of the sets in these covers, particularly the diameters of these sets. Suppose that  $q \in \mathbb{N}$ . We claim that the following hold for  $n \geq q$  and  $F \in \mathcal{E}_n$ ;

$$(4.17) \quad \text{diam } E_\lambda^{n-q}(F) \leq \beta^{3-2q},$$

$$(4.18) \quad E_\lambda^{n-q}(F) \subset \left\{ z : \text{Re}(z) \geq c_1 - \sum_{k=1}^q \beta^{3-2k} \right\},$$

and

$$(4.19) \quad E_\lambda^{n-q}(F) \subset \left\{ z : \text{Re}(z) > \max \left\{ \log \frac{2}{|\lambda|}, \beta^2 \right\} \right\}.$$

First we note that (4.19) follows from (4.5), (4.6) and (4.18).

We prove (4.17) and (4.18) by induction on  $q$ . We consider first the case that  $q = 1$ . Suppose that  $n \in \mathbb{N}$  and that  $F \in \mathcal{E}_n$ . By (4.14),  $E_\lambda^{n-1}(F)$  has diameter at most  $\beta$ . Moreover, since

$$E_\lambda^{n-1}(F) \cap W_{\nu+n-1, m'} \neq \emptyset, \quad \text{for some } m' \in \alpha_{\nu+n-1},$$

we deduce by (4.12) that  $E_\lambda^{n-1}(F) \subset \{z : \text{Re}(z) \geq c_1 - \beta\}$ . This establishes (4.17) and (4.18) in the case that  $q = 1$ .

Now, suppose that (4.17) and (4.18) have been established for  $1 \leq q \leq s$ , for some  $s \in \mathbb{N}$ . Suppose that  $n \geq s + 1$ , that  $F \in \mathcal{E}_n$  and that  $G \in \mathcal{E}_{n-1}$  is such that  $F \cap G \neq \emptyset$ . First, we deduce by (2.7), and by (4.17) and (4.19) with  $s$  in place of  $q$ , that

$$(4.20) \quad \text{diam } E_\lambda^{n-(s+1)}(F) \leq \frac{\text{diam } E_\lambda^{n-s}(F)}{\inf\{|z| : z \in E_\lambda^{n-s}(F)\}} \leq \frac{\beta^{3-2s}}{\beta^2} = \beta^{3-2(s+1)}.$$

Second, applying (4.18) with  $G$  in place of  $F$ ,  $n - 1$  in place of  $n$ , and  $s$  in place of  $q$ , we deduce that

$$E_\lambda^{(n-1)-s}(G) \subset \left\{ z : \text{Re}(z) \geq c_1 - \sum_{k=1}^s \beta^{3-2k} \right\}.$$

Since  $E_\lambda^{n-(s+1)}(F) \cap E_\lambda^{(n-1)-s}(G) \neq \emptyset$ , it follows by (4.20) that

$$E_\lambda^{n-(s+1)}(F) \subset \left\{ z : \text{Re}(z) \geq c_1 - \sum_{k=1}^s \beta^{3-2k} - \beta^{3-2(s+1)} \right\}.$$

By induction, this completes the proof of (4.17) and (4.18). We observe that it follows from (4.17) that the diameters of the sets in  $\mathcal{E}_n$  tend uniformly to zero as  $n \rightarrow \infty$ .

Suppose next that  $\epsilon > 0$ . By the definition of Hausdorff dimension, together with the observation above regarding the diameters of the sets in  $\mathcal{E}_n$ , our proof of (4.9) is complete if we can show that

$$\sum_{F \in \mathcal{E}_{n+1}} (\text{diam } F)^{1+\epsilon} \leq \sum_{F \in \mathcal{E}_n} (\text{diam } F)^{1+\epsilon}, \quad \text{for large } n,$$

or indeed if we can show that, for all sufficiently large  $n$ , we have for each  $G \in \mathcal{E}_n$  that

$$(4.21) \quad \sum_{\substack{F \in \mathcal{E}_{n+1}, \\ F \cap G \neq \emptyset}} \left( \frac{\text{diam } F}{\text{diam } G} \right)^{1+\epsilon} \leq 1.$$

Suppose that  $n \in \mathbb{N}$  and that  $G \in \mathcal{E}_n$ , in which case  $E_\lambda^n(G) = W_{\nu+n,m}$ , for some  $m \in \alpha_{\nu+n}$ . Suppose also that  $F \in \mathcal{E}_{n+1}$  intersects with  $G$ , in which case  $E_\lambda^n(F)$  intersects with  $W_{\nu+n,m}$  and is a preimage component of  $W_{\nu+n+1,m'}$ , for some  $m' \in \alpha_{\nu+n+1}$ .

We note the following two simple estimates. First, it follows from (4.13) that for each  $m' \in \alpha_{\nu+n+1}$ , there are at most  $O(e^m g_{\nu+n})$  preimage components of  $W_{\nu+n+1,m'}$  which intersect with  $W_{\nu+n,m}$ . It follows from this estimate and from (4.15), that the total number of elements of  $\mathcal{E}_{n+1}$  which intersect with  $G$  is at most

$$(4.22) \quad O \left( e^m g_{\nu+n} \log \frac{h_{\nu+n+1}}{g_{\nu+n+1}} \right) \text{ as } n \rightarrow \infty.$$

Second, it is immediate that

$$(4.23) \quad (\text{diam } E_\lambda^n(G))^{-1} = O((e^m g_{\nu+n})^{-1}) \text{ as } n \rightarrow \infty.$$

Next we estimate the distortion of  $E_\lambda^n$  in  $F \cup G$ . Suppose that  $n \geq 2$ . We deduce by (4.5) and (4.17) that

$$\text{diam } E_\lambda^m(G) < s_0, \quad \text{for } 0 \leq m < n-1.$$

It follows by (4.19) and Corollary 3.1 that  $D_G(E_\lambda^{n-1}) \leq \tau_0$ . Moreover, it follows from (2.8) that  $D_{E_\lambda^{n-1}(G)}(E_\lambda) \leq e$ . We deduce that  $D_G(E_\lambda^n) = O(1)$  as  $n \rightarrow \infty$ . By a similar argument  $D_F(E_\lambda^n) = O(1)$  as  $n \rightarrow \infty$ . Hence, since  $F \cap G \neq \emptyset$ , we have

$$(4.24) \quad D_{F \cup G}(E_\lambda^n) = O(1) \text{ as } n \rightarrow \infty.$$

Combining these estimates, we deduce from (4.14), (4.22), (4.23) and (4.24) that

$$\begin{aligned} \sum_{\substack{F \in \mathcal{E}_{n+1}, \\ F \cap G \neq \emptyset}} \left( \frac{\text{diam } F}{\text{diam } G} \right)^{1+\epsilon} &\leq O(1) \sum_{\substack{F \in \mathcal{E}_{n+1}, \\ F \cap G \neq \emptyset}} \left( \frac{\text{diam } E_\lambda^n(F)}{\text{diam } E_\lambda^n(G)} \right)^{1+\epsilon} \\ &= O \left( e^m g_{\nu+n} \log \frac{h_{\nu+n+1}}{g_{\nu+n+1}} \cdot (e^m g_{\nu+n})^{-(1+\epsilon)} \right) \\ &\leq O(g_{\nu+n}^{-\epsilon} \log h_{\nu+n+1}). \end{aligned}$$

We deduce by (4.4) that (4.21) holds, as required.  $\square$

## 5. DIMENSION RESULTS

In this section we prove Theorem 1.2, Theorem 1.3, Theorem 1.4 and Theorem 1.5. In each case we use Theorem 3.1 to show that the Hausdorff dimension is bounded below by 1, and we use Theorem 4.1 to show that the Hausdorff dimension is bounded above by 1. It is somewhat simpler to prove Theorem 1.3 before Theorem 1.2.

*Proof of Theorem 1.3.* Choose  $R \geq R_0$ , where  $R_0$  is as in the statement of Theorem 3.1, and let  $\underline{t}$  be the sequence

$$(5.1) \quad \underline{t} = 1 \ 1 \ 1 \ 2 \ 3 \ \dots \ (n-1) \ \dots$$

We deduce by Theorem 3.1 that  $\dim_H I_R(\underline{t}) \geq 1$ . Suppose that  $z \in I_R(\underline{t})$ , in which case  $R^{n-1} \leq |E_\lambda^n(z)| \leq R^n$ , for  $n \geq 2$ . It follows that  $z \in L_A(E_\lambda)$ . Hence  $I_R(\underline{t}) \subset L_A(E_\lambda)$ , and so  $\dim_H L_A(E_\lambda) \geq 1$ .

Next, for each  $p \in \mathbb{N}$ , let sequences  $(g_{p,n})_{n \in \mathbb{N}}$  and  $(h_{p,n})_{n \in \mathbb{N}}$  be defined by

$$(5.2) \quad g_{p,n} = n^{\log^+ p(n)} \text{ and } h_{p,n} = e^{pn}, \quad \text{for } n \in \mathbb{N}.$$

We deduce by Theorem 4.1 that  $\dim_H T_{g,h} \leq 1$ . Suppose that  $z \in L_A(E_\lambda)$ , in which case, by (1.5), there exist  $p, N \in \mathbb{N}$  and  $R > 1$  such that

$$n^{\log^+ p(n)} \leq |E_\lambda^n(z)| \leq R^n, \quad \text{for } n \geq N.$$

If  $p$  is sufficiently large, then  $R^n \leq e^{pn}$ , for  $n \in \mathbb{N}$ . We deduce that  $z \in T_{g,h}$ , where  $g, h$  are as defined in (5.2). It follows that  $L_A(E_\lambda) \subset T_{g,h}$ , and so  $\dim_H L_A(E_\lambda) \leq 1$ . This completes the proof of Theorem 1.3.  $\square$

*Proof of Theorem 1.2.* Choose  $R \geq R_0$ , where  $R_0$  is as in the statement of Theorem 3.1, and let  $\underline{t}$  be as defined in (5.1). Recall that  $\dim_H I_R(\underline{t}) \geq 1$ . We deduce Theorem 1.2 from Theorem 1.3 and the fact that  $I_R(\underline{t}) \subset L_U(E_\lambda) \subset L_A(E_\lambda)$ .  $\square$

*Proof of Theorem 1.4.* Choose  $R \geq R_0$ , where  $R_0$  is as in the statement of Theorem 3.1, and let  $\underline{t}'$  be the sequence

$$\underline{t}' = 1 \ 4 \ 9 \ \dots \ (n+1)^2 \ \dots$$

We deduce by Theorem 3.1 that  $\dim_H I_R(\underline{t}') \geq 1$ . Suppose that  $z \in I_R(\underline{t}')$ , in which case  $R^{(n+1)^2} \leq |E_\lambda^n(z)| \leq R^{(n+1)^2+1}$ , for  $n \geq 0$ . It follows that  $z \in M_A(E_\lambda)$ . Hence  $I_R(\underline{t}') \subset M_A(E_\lambda)$ , and so  $\dim_H M_A(E_\lambda) \geq 1$ .

Next, for each  $p \in \mathbb{N}$ , let sequences  $(g'_{p,n})_{n \in \mathbb{N}}$  and  $(h'_{p,n})_{n \in \mathbb{N}}$  be defined by

$$(5.3) \quad g'_{p,n} = e^{n \log^+ p(n)} \text{ and } h'_{p,n} = \exp(e^{pn}), \quad \text{for } n \in \mathbb{N}.$$

We deduce by Theorem 4.1 that  $\dim_H T_{g',h'} \leq 1$ .

Suppose that  $z \in M_A(E_\lambda)$ , in which case, by (1.6), there exist  $p, N \in \mathbb{N}$  and  $R > 1$  such that

$$e^{n \log^+ p(n)} \leq |E_\lambda^n(z)| \leq \exp(e^{pn}), \quad \text{for } n \geq N.$$

We deduce that  $z \in T_{g',h'}$ , where  $g', h'$  are as defined in (5.3). It follows that  $M_A(E_\lambda) \subset T_{g',h'}$ , and so  $\dim_H M_A(E_\lambda) \leq 1$ . This completes the proof of Theorem 1.4.  $\square$

Finally we prove Theorem 1.5.



*Proof of Theorem 1.5.* Suppose that  $\lambda \neq 0$ ,  $R > 1$  and  $\underline{t} = t_0 t_1 t_2 \dots$  is an escaping annular itinerary for the function  $E_\lambda$ . For each  $p \in \mathbb{N}$  we define sequences  $(g_{p,n})_{n \in \mathbb{N}}$  and  $(h_{p,n})_{n \in \mathbb{N}}$  by  $g_{p,n} = R^{t_n}$ , and  $h_{p,n} = R^{t_n+1}$ . We deduce by Theorem 4.1 that  $\dim_H T_{g,h} \leq 1$ . The first part of the theorem follows since  $I_R(\underline{t}) \subset T_{g,h}$ .

The second part of Theorem 1.5 is an immediate consequence of this, together with Theorem 3.1.  $\square$

## 6. THE UNIFORMLY SLOWLY ESCAPING SET

In this section, for a general transcendental entire function  $f$ , we give two results on the set  $L_U(f)$ . First, we give a necessary and sufficient condition for  $L_U(f)$  to be non-empty. Here  $m(r, f) = \min_{|z|=r} |f(z)|$  denotes the *minimum modulus* of  $f$ , for  $r > 0$ .

**Theorem 6.1.** *Suppose that  $f$  is a transcendental entire function. Then*

$$L_U(f) \cap J(f) \neq \emptyset$$

*if and only if there exist positive constants  $c$  and  $r_0$ , and  $d > 1$  such that*

$$(6.1) \quad \text{for all } r \geq r_0 \text{ there exists } \rho \in (r, dr) \text{ such that } m(\rho, f) \leq c.$$

*Moreover, if  $L_U(f) \neq \emptyset$ , then  $L_U(f) \cap J(f) \neq \emptyset$ .*

**Remark 4.** The condition (6.1) was used in [21], also in relation to points tending to infinity at a specified rate. As observed in [21], this condition holds whenever  $f$  is bounded on a path to infinity. It is well-known that this is the case for functions in the class  $\mathcal{B}$  in particular.

Second, we show that when  $L_U(f)$  is not empty, it has a number of familiar properties which show that, in general, this is a dynamically interesting set. We say that a set  $S$  is *completely invariant* if  $z \in S$  implies that  $f(z) \in S$  and also that  $f^{-1}(\{z\}) \subset S$ .

**Theorem 6.2.** *Suppose that  $f$  is a transcendental entire function, and that  $L_U(f) \neq \emptyset$ . Then the following all hold.*

- (i)  $L_U(f)$  is completely invariant;
- (ii) If  $U$  is a Fatou component of  $f$  and  $U \cap L_U(f) \neq \emptyset$ , then  $U \subset L_U(f)$ ;
- (iii)  $L_U(f)$  is dense in  $J(f)$  and  $J(f) = \partial L_U(f)$ .

**Remark 5.** An example of a transcendental entire function with a Fatou component contained in  $L_U(f)$  is the function  $f(z) = 2 - \log 2 + 2z - e^z$ , given by Bergweiler [2]. It is straightforward to deduce from the arguments in [2] that  $f$  has a Baker domain in  $L_U(f)$  and also wandering Fatou components in  $L_U(f)$ . We refer to [1] for definitions.

The proof of Theorem 6.1 requires the following [21, Theorem 2].

**Lemma 6.1.** *Suppose that  $f$  is a transcendental entire function. Then  $f$  has the property that, for all positive sequences  $(a_n)_{n \in \mathbb{N}}$  such that  $a_n \rightarrow \infty$  as  $n \rightarrow \infty$  and  $a_{n+1} = O(M(a_n, f))$  as  $n \rightarrow \infty$ , there exist  $\zeta \in J(f)$  and  $C > 1$  such that*

$$a_n \leq |f^n(\zeta)| \leq Ca_n, \quad \text{for } n \in \mathbb{N},$$

*if and only if there exist positive constants  $c$  and  $r_0$ , and  $d > 1$  such that (6.1) holds.*

*Proof of Theorem 6.1.* Suppose that  $f$  is a transcendental entire function. If there exist positive constants  $c$  and  $r_0$ , and  $d > 1$  such that (6.1) holds, then it follows immediately from Lemma 6.1 that  $L_U(f) \cap J(f) \neq \emptyset$ .

The other direction proceeds very similarly to the proof of the corresponding direction in [21, Theorem 2]. We give some details for completeness. Suppose that there do not exist positive constants  $c$  and  $r_0$ , and  $d > 1$  such that (6.1) holds. Then there exists a sequence of annuli  $A(r_n, R_n)$ , where  $0 < r_n < R_n$ , such that  $r_n \rightarrow \infty$  as  $n \rightarrow \infty$ ,  $R_n/r_n \rightarrow \infty$  as  $n \rightarrow \infty$  and

$$m(r, f) > 1, \quad \text{for } r_n < r < R_n, \quad n \in \mathbb{N}.$$

As shown in the proof of [21, Theorem 2], it follows that there exists  $\delta \in (0, 1)$  and  $N \in \mathbb{N}$  such that

$$(6.2) \quad m(r, f) > M(r, f)^\delta, \quad \text{for } 2r_n < r < \frac{1}{2}R_n, \quad n \geq N.$$

We shall deduce that  $L_U(f) = \emptyset$ . Suppose, by way of contradiction, that there exists  $z \in L_U(f)$ . It follows from (1.3) that there exists  $C > 0$  and  $N \in \mathbb{N}$  such that

$$(6.3) \quad f^n(z) \neq 0 \text{ and } \left| \frac{f^{n+1}(z)}{f^n(z)} \right| \leq C, \quad \text{for } n \geq N.$$

We deduce that, for infinitely many values of  $n \in \mathbb{N}$ , there exists  $p(n) \in \mathbb{N}$  such that  $f^n(z) \in A(2r_{p(n)}, \frac{1}{2}R_{p(n)})$ . It follows by (6.2) that

$$\left| \frac{f^{n+1}(z)}{f^n(z)} \right| > \frac{M(|f^n(z)|, f)^\delta}{|f^n(z)|}, \quad \text{for infinitely many values of } n \in \mathbb{N}.$$

Since, as is well-known,

$$\frac{M(r, f)^\delta}{r} \rightarrow \infty \text{ as } r \rightarrow \infty,$$

this is a contradiction to (6.3). We deduce that  $L_U(f) \cap J(f) = \emptyset$  if and only if (6.1) holds. Finally, if  $L_U(f) \neq \emptyset$ , then (6.1) holds, and so  $L_U(f) \cap J(f) \neq \emptyset$ . This completes the proof of Theorem 6.1.  $\square$

In order to prove Theorem 6.2, we require the following well-known distortion lemma; see, for example, [1, Lemma 7].

**Lemma 6.2.** *Suppose that  $f$  is a transcendental entire function, and that  $U \subset I(f)$  is a simply connected Fatou component of  $f$ . Suppose that  $K$  is a compact subset of  $U$ . Then there exist  $C > 1$  and  $N_0 \in \mathbb{N}$  such that*

$$\frac{1}{C}|f^n(z)| \leq |f^n(w)| \leq C|f^n(z)|, \quad \text{for } w, z \in K, n \geq N_0.$$

We also use the following, which is a special case of [21, Lemma 10]. We say that a set  $S$  is *backwards invariant* if  $z \in S$  implies that  $f^{-1}(\{z\}) \subset S$ .

**Lemma 6.3.** *Suppose that  $f$  is a transcendental entire function, and that  $E \subset \mathbb{C}$  contains at least three points. Suppose also that  $E$  is backwards invariant under  $f$ , that  $\text{int } E \cap J(f) = \emptyset$ , and that every component of  $F(f)$  that meets  $E$  is contained in  $E$ . Then  $\partial E = J(f)$ .*

*Proof of Theorem 6.2.* Suppose that  $f$  is a transcendental entire function, and that  $L_U(f) \neq \emptyset$ , in which case, by Theorem 6.1,  $L_U(f) \cap J(f) \neq \emptyset$ . First we observe that part (i) of the theorem follows immediately from the definition of  $L_U(f)$ .

For part (ii) of the theorem, suppose that  $U$  is a Fatou component of  $f$ , such that  $U \cap L_U(f) \neq \emptyset$ . It follows by normality that  $U \subset I(f)$ .

Suppose that  $U$  is multiply connected in which case [22, Theorem 1.2] we have that  $U \subset A(f)$ . However, it is known [3] that if  $z \in A(f)$  then

$$\frac{\log \log |f^n(z)|}{n} \rightarrow \infty \text{ as } n \rightarrow \infty,$$

in which case  $z \notin L_U(f)$ . We deduce that  $U$  is simply connected.

Suppose that  $z \in U \cap L_U(f)$ . Then there exist  $N \in \mathbb{N}$ ,  $R > 1$  and  $0 < C_1 < C_2$  such that  $C_1 R^n \leq |f^n(z)| \leq C_2 R^n$ , for  $n \geq N$ . Suppose that  $K$  is a compact subset of  $U$  containing  $z$ . Then, by Lemma 6.2 there exist  $C > 1$  and  $N_0 \in \mathbb{N}$  such that

$$\frac{C_1}{C} R^n \leq |f^n(w)| \leq C_2 C R^n, \text{ for } w \in K, n \geq \max\{N, N_0\}.$$

Hence  $K \subset L_U(f)$ , and so  $U \subset L_U(f)$ . This completes the proof of part (ii).

For part (iii) of the theorem, we note that  $L_U(f) \cap J(f)$  is an infinite set, since for each  $z \in L_U(f) \cap J(f)$  at least one of the points  $z, f(z)$  or  $f^2(z)$  must have an infinite backwards orbit. It is known [1, Theorem 4] that the set of repelling periodic points of  $f$  is dense in  $J(f)$ . Since, by definition,  $L_U(f)$  contains no periodic points,  $\text{int } L_U(f) \subset F(f)$ . The result follows by part (i) and part (ii), and by Lemma 6.3 applied with  $E = L_U(f) \cap J(f)$  and then with  $E = L_U(f)$ . This completes the proof of Theorem 6.2.  $\square$

## 7. RESULTS OF KARPIŃSKA AND URBAŃSKI

As mentioned in the introduction, Karpińska and Urbański [16] studied the size of various subsets of  $I(E_\lambda)$ . For integers  $k \geq 0$  and  $l \geq k$ , and  $\epsilon > 0$ , they defined sets

$$D_\epsilon^{k,l} = \left\{ z \in I(E_\lambda) : \text{Re}(E_\lambda^n(z)) > q, \text{ for } n \geq k, \text{ and } |\text{Im}(E_\lambda^n(z))| \leq \frac{|E_\lambda^n(z)|}{(\log |E_\lambda^n(z)|)^\epsilon}, \text{ for } n \geq l \right\},$$

where  $q$  is fixed and large. Their main result is the following.

**Theorem 7.1.** *For every  $\epsilon > 0$  and all integers  $0 \leq k \leq l$ ,*

$$\dim_H D_\epsilon^{k,l} = 1 + \frac{1}{1 + \epsilon}.$$

In this section we show that the sets  $D_\epsilon^{k,l}$  lie in the *fast* escaping set of  $E_\lambda$ . First, we define the domain  $V = \{z : \text{Re}(z) > \frac{1}{2}|z|\}$ . Note that

$$(7.1) \quad |E_\lambda(z)| = |\lambda|e^{\text{Re}(z)} > |\lambda|e^{\frac{1}{2}|z|} = M\left(\frac{1}{2}|z|, E_\lambda\right), \text{ for } z \in V.$$

Suppose that  $\epsilon > 0$  and  $0 \leq k \leq l$ , and that  $z \in D_\epsilon^{k,l}$ . It follows from the definition of  $D_\epsilon^{k,l}$  that there exists  $N \in \mathbb{N}$  such that  $E_\lambda^{n+N}(z) \in V$ , for  $n \geq 0$ . Set  $R = \frac{1}{2}|E_\lambda^N(z)|$ , and define

$$\mu(r) = \frac{1}{2}M(r, E_\lambda), \text{ for } r \geq 0.$$

We may assume that  $R$  is sufficiently large that  $\mu(r) > r$ , for  $r \geq R$ . It follows from (7.1) that

$$|E_\lambda^n(E_\lambda^N(z))| > \mu^n(R), \quad \text{for } n \geq 0,$$

and so, by [22, Theorem 2.9], that  $z \in A(E_\lambda)$ , as required.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, THE OPEN UNIVERSITY, WALTON HALL, MILTON KEYNES MK7 6AA, UK

*E-mail address:* david.sixsmith@open.ac.uk